# Positive Solution of Fractional Differential Equations with Integral Boundary Condition

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**Abstract:** This present work concerns the study of a class of nonlinear fractional differential equations with an integral condition, by the help of some fixed point theorems. We establish the uniqueness result by the Banach contraction principle and to prove the existence of positive solution we use a cone fixed point theorem due to Guo-Krasnoselskii by introducing height functions of the nonlinear term on some bounded sets and considering integrations of these height functions. Two examples are also included to illustrate our results.

**Keywords:** Cone, fractional differential equations, fixed-point theorem, Integral condition, Riemann-Liouville fractional derivative.
**Mathematics Subject Classifications:** 34B10, 34B15.

# Introduction

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Fractional boundary value problems have been widely studied in the last decades and many monographs and books are devoted to this subject we refer to [1,4,5,7-13,15-19,21,23-24] and their references.

Fractional and ordinary boundary value problems with integral conditions have been investigated by many authors see $[2,3,6,14,20,...]$. The history of fractional calculus can be traced back to the 17th century, when the German mathematician Gottfried Leibniz first mentioned the concept of fractional differentiation in a letter to his colleague Johann Bernoulli. However, the development of fractional calculus as a field of study actually began in the 19th century, with the work of several mathematicians, including Augustin-Louis Cauchy, Liouville, and Riemann. In the early 20th century, the French mathematician Paul Lévy used fractional calculus to model random processes, and it was subsequently used in the study of fractals and other areas of mathematics. We can cite the paper [20], where Wenxia Wang studied the following fractional integral boundary value problem (BVP) with a parameter $\mu$.

$$\begin{cases} D_{0^+}^{\alpha}x(t)+f(t,x(t))=0, & t\in(0,1). \\ x'(0)=0, \quad x(1)=\mu\int_0^1 x(s)ds, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha$, $1 < \alpha < 2$, $f \in C([0,1] \times \mathsf{R}, \mathsf{R}_+)$ and $\mu > 0$.

In [22], the authors obtained the existence of positive solutions of the following singular fractional differential equations with infinite-point boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha} x(t) + q(t) f(t, x(t)) = 0, \quad t \in (0,1). \\ x(0) = x'(0) = \ldots x^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} x(1) = \sum\limits_{i=1}^{\infty} \alpha_i x(\xi_i), \end{cases}$$

where $\alpha > 2$, $n - 1 < \alpha \le n$, $\beta \in [1, \alpha - 1]$ is a fixed number, $\alpha_i \ge 0$, $0 < \xi_1 < \ldots < \xi_i < \ldots < 1$, and $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville derivatives.

Motivated by the above work, we investigate the following integral boundary value problems of fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1). \\ u(0) = u'(0) = 0, \quad u(1) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s, u(s)) ds, \end{cases} \qquad (1.1)$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville differential operator, of order $\alpha$, $2 < \alpha \le 3$ and $0 < \beta < \alpha$. $f$, $g \in C([0,1] \times (0, +\infty), [0, +\infty)), g(t,u)$ is nondecreasing on $u$ for any $t \in [0,1]$.

The organization of this paper is as follows. In section 2, we provide necessary background. Section 3 treats the uniqueness of solution by using the Banach contraction principle. Section 4 is devoted to the existence of positive solution on a cone, then we give some examples. We achieve the paper with two examples.

# Preliminaries

In this section, we introduce some basic definitions and lemmas

**Definition 1 :** [16] *The fractional integral*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

*where* $\alpha > 0$, *is called Riemann-Liouville fractional integral of order* $\alpha$ *of a function* $f : (0, +\infty) \to \mathsf{R}$ *and* $\Gamma(.)$ *is the gamma function.*

**Definition 2:** [16] *The Riemann-Liouville fractional derivative of order* $\alpha > 0$, *of a continuous function* $f : (0, +\infty) \to \mathsf{R}$ *is given by*

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha+1} f(s) ds.$$

$\Gamma(.)$ *is the gamma function, provided that the right side is point-wise defined on* $(0, +\infty)$ *and* $n = [\alpha] + 1$, $[\alpha]$ *stands for the integer less than* $\alpha$.

**Lemma 3:** $[10]$ *Let* $\alpha,\beta>0,\ \ f\in L(0,1),$ *then*

$$I_{0^+}^{\alpha}I_{0^+}^{\beta}f(t)=I_{0^+}^{\alpha+\beta}f(t).$$

**Lemma 4:** $[10]$ *Assume that* $u\in C(0,1)\cap L^1(0,1)$ *with a frational derivative of order* $\alpha>0$ *that belongs to* $C(0,1)\cap L^1(0,1).$ *Then*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\ldots+c_nt^{\alpha-n},$$

*for some* $c_i\in \mathbb{R},\ i=1,2,\ldots,n;\ n=[\alpha]+1.$

**Lemma 5:** $[22]$ *If* $\alpha,\beta>0,$ *then*

$$I^{\alpha}t^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}t^{\beta+\alpha-1}.$$

Let $E$ be the Banach space of continuous functions $C[0,1],$ endowed with the norm $\|u\|_E=\max_{t\in[0,1]}|u(t)|.$

**Lemma 6:** *The unique solution of the problem* $(1.1)$ *is given by*

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)\,f(s,u(s))ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)g(s,u(s))ds,$$

*where*

$$G(t,s)=\begin{cases}t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1.\end{cases}$$

$$H(t,s)=t^{\alpha-1}(1-s)^{\alpha-\beta-1},\ \ t,s\in[0,1]$$

**Proof:** By $Lemma\ 4$ we can see that

$$u(t)=-I_{0^+}^{\alpha}y(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+C_3t^{\alpha-3}.$$

From $u(0)=u'(0)=0$ we get $C_3=C_2=0.$ And, from $u(1)=\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}g(s,u(s))ds,$ we deduce that

$$C_1=\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}g(s,u(s))ds+I_{0^+}^{\alpha}y(1).$$

Then

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t-(t-s)^{\alpha-1}f(s,u(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}f(s,u(s))ds$$
$$+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t t^{\alpha-1}(1-s)^{\alpha-\beta-1}g(s,u(s))ds.$$

So

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\left[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]f(s,u(s))ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-1}f(s,u(s))ds$$
$$+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t t^{\alpha-1}(1-s)^{\alpha-\beta-1}g(s,u(s))ds.$$

And, that is equivalente to

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)\,f(s,u(s))ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)g(s,u(s))ds,$$

where $G(t,s)$ is defined by $(2.2)$. The proof is complete.

Now we give the properties of the Green function.

**Lemma 7:** *The function* $G(t,s)$ *defined by* $(2.2)$ *satisfies the following properties*

$(i)$ $G(t,s)\geq 0$ *and* $G(t,s)\in C([0,1]\times[0,1],\mathbb{R}_+)$ .

$(ii)\max\limits_{t\in[0,1]} G(t,s)=G_1(s)$

$(iii)$ *If* $t,s\in[\tau,1],\quad \tau>0,$ *then*

$$\tau^{\alpha-1}G_1(s)\leq G(t,s)\leq\frac{1}{\tau}G_1(s),$$

*where* $G_1(s)=s(1-s)^{\alpha-1}$.

**Proof:** $(i)$ The continuity of $G$ is easily checked. For $0\leq t\leq s\leq 1$, it is obvious that

$$G(t,s)=(1-s)^{\alpha-1}t^{\alpha-1}\geq 0.$$

In the case, $0\leq s\leq t\leq 1,$ we have

$$G(t,s)=\left[(1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}\right]=(t-ts)^{\alpha-1}-(t-s)^{\alpha-1}\geq 0.$$

$(ii)$ is easily checked.

$(iii)$ If $0\leq t\leq s\leq 1,$

$$G(t,s)=(1-s)^{\alpha-1}t^{\alpha-1}\leq G_1(s).$$

If $0\leq s\leq t\leq 1$ , we have

$$G(t,s)=\left[(1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}\right],$$

then

$$G(t,s)\leq\frac{1}{s}G_1(s),\quad \forall s,t\in[0,1].$$

Consequently

$$G(t,s)\leq\frac{1}{\tau}G_1(s),\quad \forall s\in[\tau,1],\ t\in[0,1].$$

Now, we look for lower bounds of $G(t,s)$. If $0\leq t\leq s\leq 1,$

$$G(t,s)=t^{\alpha-1}(1-s)^{\alpha-1}\geq t^{\alpha-1}s(1-s)^{\alpha-1},$$

then

$$G(t,s)\geq t^{\alpha-1}G_1(s),\quad \forall s,t\in[0,1].$$

If $0\leq s\leq t\leq 1$ , we have

$$G(t,s)=(1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}\geq 0,$$

and

$$(1-s)^{\alpha-1}t^{\alpha-1}(1-s)-(t-s)^{\alpha-1}\geq 0,$$

$$G(t,s)\geq t^{\alpha-1}G_1(s),\quad \forall s,t\in[0,1].$$

Finally,

$G(t,s) \geq \tau^{\alpha-1} G_1(s)$, for $t,s \in [\tau,1]$.

The proof is complete.

To use the fixed point theorem, according to *Lemma* 6, we define the operator $T$ as

$$Tu(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 G(t,s)\, f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 H(t,s) g(s,u(s))ds.$$

Then, we have the following *Lemma*.

**Lemma 8:** *The operator* $T \,:\, E \to E$, *is completely continuous.*

**Proof:** 1) $T$ is continuous.

From the continuity of $f$, $H$ and $G$ , we conclude that $T$ is continous operator

2) Let $B_r = \{u \in E \,:\, \|u\|_E \leq r\}$ a bounded subset. we will prove that $T(\Omega \cap B_r)$ is relatively compact, $\Omega = \{u \in E \,:\, \|u\|_E < m\}$.

$(i)$ $T(\Omega \cap B_r)$ is uniformly bounded.

Then for any $t \in B_r$, there exists a constant $M$, such that $f(s,u)$, $g(s,u) \leq M$, for some $u \in \Omega \cap B_r$, we have:

$$|Tu(t)| \leq \frac{1}{\tau} \frac{1}{\Gamma(\alpha)} \int_0^1 G_1(s)\, f(s,u(s))ds + \frac{1}{\tau} \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 G_1(s) g(s,u(s))ds.$$

Then,

$$\|Tu\|_E \leq \frac{M}{\tau}\left( \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta)} \right) \int_0^1 G_1(s)ds,$$

then, $T(\Omega \cap B_r)$ uniformly bounded.

$(ii)$ $T(\Omega \cap B_r)$ is equicontinuous.

Because $G(t,s)$ is continuous on $[0,1] \times [0,1]$, $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$, ( likewise for $H(t,s)$ ). Thus for any $\varepsilon_1$, $\varepsilon_2$, there existe $\delta > 0$ such that $|G(t_1,s) - G(t_2,s)| < \varepsilon_1$ and $|H(t_1,s) - H(t_2,s)| < \varepsilon_2$, let $u \in \Omega \cap B_r$ , $\forall\ t_1$, $t_2 \in [0,1]$; $t_1 < t_2$, and $|t_1 - t_2| < \delta$ , one has:

$$|Tu(t_2) - Tu(t_1)| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 |G(t_2,s) - G(t_1,s)|\, f(s,u(s))ds$$

$$+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 |H(t_2,s) - H(t_1,s)|\, g(s,u(s))ds.$$

$$|Tu(t_2) - Tu(t_1)| \leq M\left( \frac{\varepsilon_1}{\Gamma(\alpha)} + \frac{\varepsilon_2}{\Gamma(\alpha-\beta)} \right).$$

Consequently, $T(\Omega \cap B_r)$ is equicontinuous. From Arzela-Ascoli theorem, we deduce that $T$ completely continuous operator.

Uniqueness solution

In this section, we prove the uniqueness result by the Banach contraction principle.

**Theorem 9:** *Assume that there are* $L_1, L_2 > 0$ *such that*

$$|f(t,u) - f(t,v)| \leq L_1 \|u - v\|,$$
$$|g(t,u) - g(t,v)| \leq L_2 \|u - v\|,$$

$\forall u, v \in \mathbb{R}_+,\ t \in [0,1]$,

*and if*

$$C = \frac{1}{\tau}\left(\frac{L_1}{\Gamma(\alpha)} + \frac{L_2}{\Gamma(\alpha - \beta)}\right)\int_0^1 G_1(s)ds < 1, \quad \tau \in [0,1].$$

*Then the boundary value broblem* $(1.1)$, *has a unique solution in* $E$

**Proof:** We shall use the Banach contraction principle to prove that the operator $T$ defined by $(2.3)$ has a fixed point. Now we will prove that $T$ is a contraction. Let $u, v \in E$, we get

$$|Tu(t) - Tv(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)|f(s,u(s)) - f(s,v(s))|\,ds$$
$$+ \frac{1}{\Gamma(\alpha - \beta)}\int_0^1 H(t,s)|g(s,u(s)) - g(s,v(s))|\,ds.$$

So, we can obtain

$$|Tu(t) - Tv(t)| \leq \frac{1}{\tau}\left(\frac{L_1}{\Gamma(\alpha)} + \frac{L_2}{\Gamma(\alpha - \beta)}\right)\int_0^1 G_1(s)ds\,\|u - v\|.$$

By using

$$C = \frac{1}{\tau}\left(\frac{L_1}{\Gamma(\alpha)} + \frac{L_2}{\Gamma(\alpha - \beta)}\right)\int_0^1 G_1(s)ds < 1.$$

Obviously, we have

$$\|Tu - Tv\|_E \leq C\|u - v\|_E,$$

so, the contraction principle ensures the uniqueness of a solution for the fractional boundary value problem $(1.1)$. This finishes the proof.

# Existence of positive solution

In this section we investigate the positivity of solutions for the *fractional boundary value problem* $(1.1)$ , for this we make the following hypotheses.

$(Q_1)$ $f, g \in C((0,1) \times (0,+\infty), [0,+\infty))$.

$(Q_2)$ $\int_0^1 G_1(s)ds > 0$.

$(Q_3)$ For any positive numbers $r_1 < r_2$, there exists a continuous function $p_{r_1,r_2} : (0,1) \to [0,+\infty)$ and $q_{r_1,r_2} : (0,1) \to [0,+\infty)$ such that

$$f(t,u) \leq p_{r_1,r_2}(t), \quad 0 < t < 1,\ t^\alpha r_1 \leq u \leq r_2.$$

Let $E = C[0,1]$, so that $E$ is a Banach space endowed with the norm $\|u\| = \max\limits_{0 \le t \le 1} |u(t)|$.

The main result of this section is the following well-known Guo-Krasnosel'skii fixed point theorem on cone.

**Theorem 10:** $[9]$ *Let* $E$ *be a Banach space, and let* $K \subset E$, *be a cone. Assume* $\Omega_1, \Omega_2$ *are open subsets of* $E$ *with* $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, *and let*
$\mathbf{A} : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to K$,
*be a completely continuous operator. In addition suppose either*
$(i)$ $\|\mathbf{A}u\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathbf{A}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
$(ii)$ $\|\mathbf{A}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathbf{A}u\| \le \|u\|$, $u \in K \cap \partial\Omega_2$,
*holds. Then* $\mathbf{A}$ *has a fixed point in* $K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$.

**Definition 11:** *A function* $u(t)$ *is called positive solution for the boundary value problem* $(1.1)$ *if* $u(t) \ge 0$, $\forall t \in [0,1]$ .

**Lemma 12:** *Let* $u \in E$ *, then the solution* $u$ *of the fractional boundary value problem* $(1.1)$ *is nonnegative and satisfies*

$$\min_{\tau \in [t,1]} u(t) \ge \tau^{\alpha} \|u\|_E .$$

**Proof:** Let $u \in E$, it is obvious that $u(t)$ is nonnegative, $t \in [0,1]$. From *Lemma* 6 and 7 , we have

$$\|u\|_E \le \frac{1}{\tau}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s) f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s) g(s,u(s))ds\right).$$

Hence

$$\tau\|u\|_E \le \frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s) f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s) g(s,u(s))ds.$$

On the other hand, for all $t \in [\tau,1]$, we obtain

$$u(t) \ge \tau^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s) f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s) g(s,u(s))ds\right).$$

Therefore, we have

$$\min_{t \in [\tau,1]} u(t) \ge \tau^{\alpha} \|u\|_E .$$

The proof is complete.

Let $K$ be the cone of nonnegative function in $C[0,1]$ with the following form

$$K = \left\{u \in E,\ u(t) \ge \tau^{\alpha} \|u\|_E ,\ t \in [\tau,1]\right\}.$$

$K$ is a nonempty closed and convex subset of $E$, hence it is a cone.

Denote $\Omega(r) = \left\{u \in K : \|u\|_E < r\right\}$ and $\partial\Omega(r) = \left\{u \in K : \|u\|_E = r\right\}$, for $r > 0$.

**Lemma 13:** *Soppose that* $(Q_1)-(Q_3)$ *hold and* $0 < r_1 < r_2$. *Then* $T : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to K$, *is completely continous*

**Proof:** For any $u \in \overline{\Omega(r_2)} \setminus \Omega(r_1)$, it follows from *Lemma* 7 that

$$Tu(t) \le \frac{1}{\tau}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\, f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\right).$$

On the other hand, for all $t \in [\tau,1]$, we obtain

$$Tu(t) \ge \tau^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\, f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\right).$$

Therefore, we have

$$Tu(t) \ge \tau^{\alpha}\|Tu\|_E,\ \ 0 \le \tau \le t \le 1.$$

Noticing the continuity of $G(t,s)$ and $(Q_1)-(Q_3)$, it i seasy to see that $T$ is continous in $\overline{\Omega(r_2)} \setminus \Omega(r_1)$. Next, we show $T$ is compact. For any $u \in \overline{\Omega(r_2)} \setminus \Omega(r_1)$, we have

$$|Tu(t)| \le \frac{1}{\tau}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\, f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\right].$$

Then,

$$\|Tu\|_E \le \frac{M}{\tau}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta)}\right)\int_0^1 G_1(s)ds,$$

where, there exists a constant $M$ , such that $f(s,u)$, $g(s,u) \le M$. Then, $T\left(\overline{\Omega(r_2)} \setminus \Omega(r_1)\right)$ is uniformly bounded.

Because $G(t,s)$ is continuous on $[0,1]\times[0,1]$, $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, ( likewise for $H(t,s)$ ) . Thus for any $\varepsilon_1$, $\varepsilon_2$, there existe $\delta > 0$ such that $|G(t_1,s) - G(t_2,s)| < \varepsilon_1$ and $|H(t_1,s) - H(t_2,s)| < \varepsilon_2$, if $|t_1 - t_2| < \delta$, and $(t_1,s)$, $(t_2,s) \in [0,1]\times[0,1]$. Then, for any $u \in \overline{\Omega(r_2)} \setminus \Omega(r_1)$ and $t_1, t_2 \in [0,1]$ such that $|t_1 - t_2| < \delta$ , we have

$$|Tu(t_2) - Tu(t_1)| \le \frac{1}{\Gamma(\alpha)}\int_0^1 |G(t_2,s) - G(t_1,s)|\, f(s,u(s))ds$$

$$+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 |H(t_2,s) - H(t_1,s)|\, g(s,u(s))ds.$$

$$|Tu(t_2) - Tu(t_1)| \le M\left(\frac{\varepsilon_1}{\Gamma(\alpha)} + \frac{\varepsilon_2}{\Gamma(\alpha-\beta)}\right)$$

We can see that the functions in $T\left(\overline{\Omega(r_2)} \setminus \Omega(r_1)\right)$ are equicontinuous. So, $T\left(\overline{\Omega(r_2)} \setminus \Omega(r_1)\right)$ is relatively compact in $E$. Thereby, $T$ is compact in $\overline{\Omega(r_2)} \setminus \Omega(r_1)$, and thus $T : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to K$ is completely continuous.

We introduce the following height functions to control the growth of the nonlinear term $f(t,x)$:

$$\varphi(t,r)=\max\left\{f(t,u(t))\,:\,t^{\alpha}r\le u\le r\right\},\ 0<t<1,\ r>0.$$
$$\psi(t,r)=\min\left\{f(t,u(t))\,:\,t^{\alpha}r\le u\le r\right\},\ 0<t<1,\ r>0.$$

**Theorem 14:** *Suppose that* $(Q_1)-(Q_3)$ *hold and there exist two positive numbers* $a<b$ *such that one of the following conditions is satisfied:*

$(A_1)$ : $a\le\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\psi(s,a)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds<+\infty$

*and,* $\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\varphi(s,b)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\le b,$

$(A_2)$ : $\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\varphi(s,a)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\le a$

*and,* $b\le\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\psi(s,b)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds<+\infty,$

*where,* $g(t,u)$ *is nondecreasing on* $u$ *for any* $t\in[0,1]$.

*Then the boundary value problem* $(1.1)$ *has at least one strictly increasing positive solution* $u^*\in K$ *such that* $a\le\|u^*\|\le b$.

**Proof:** Without loss of generality, we only prove $(A_1)$.

If $u\in\partial\Omega(a)$, then $\|u\|=a$ and $t^{\alpha}a\le u(t)\le a$, $0\le t\le 1$. By the definition of $\psi(t,a)$ we know that

$f(t,u(t))\ge\psi(t,a)$.

By *Lemma* 7 we have that

$$\|Tu\|=\max_{t\in[0,1]}\frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)f(s,u(s))ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)g(s,u(s))ds.$$

$$\|Tu\|\ge\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\psi(s,a)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\ge a=\|u\|.$$

If $u\in\partial\Omega(b)$, then $\|u\|=b$ and $t^{\alpha}b\le u(t)\le b$, $0\le t\le 1$. By the definition of $\varphi(t,a)$ we know that

$f(t,u(t))\le\varphi(t,a)$.

By *Lemma* 7 we have that

$$\|Tu\|=\max_{t\in[0,1]}\frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)f(s,u(s))ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)g(s,u(s))ds.$$

$$\|Tu\|\le\left(\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)f(s,u(s))ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\right).$$

$$\|Tu\|\le\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\varphi(s,b)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds\le b=\|u\|.$$

By *Theorem* 10 (Guo-Krasnosel'skii fixed point theorem), $T$ has a fixed point $u^*\in\overline{\Omega(b)}\setminus\Omega(a)$, From *Section* 2 we know that $u^*$ is a solution of $(1.1)$ and $a\le\|u^*\|\le b$.

Because $u^*(t) \geq t^\alpha \|u^*\| \geq t^\alpha a, \quad 0 \leq t \leq 1,$ we get that $u^*$ is a positive solution for $(1.1)$. And, we have

$$\begin{aligned}(u^*)'(t) &= (Tu^*)'(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 \frac{\partial}{\partial t} G(t,s) f(s,u(s))ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 \frac{\partial}{\partial t} H(t,s) g(s,u(s))ds > 0,\end{aligned}$$

which implies that $u^*$ is a strictly increasing positive solution. The proof is completed.

# Examples

In order to illustrate our results, we give the following examples.

**Example 15:** *Consider the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \frac{t^2}{m} u(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} \left(1 + \frac{u(s)}{n}\right) ds, \end{cases} \qquad (P_1)$$

*Let,*

$$\alpha = \frac{7}{3}, \ \beta = \frac{2}{3},$$

*and,*

$$f(t,u(t)) = \frac{t^2}{m} u(t), \quad g(t,u(t)) = 1 + \frac{u(t)}{n}, \quad m,n > 0.$$

*Then,*

$$|f(t,u) - f(t,v)| \leq \frac{t^2}{m}\|u-v\|,$$

$$|g(t,u) - g(t,v)| \leq \frac{1}{n}\|u-v\|, \ \ \forall u,v \in \mathrm{R}_+, \ t \in [0,1],$$

*and,*

$$C = \frac{1}{\tau}\left(\frac{L_1}{\Gamma(\alpha)} + \frac{L_2}{\Gamma(\alpha-\beta)}\right)\int_0^1 G_1(s)ds < 1, \quad \tau \in [0,1].$$

*Hence, by* Theorem 9*, the boundary value problem* $(P_1)$ *has a unique solution in* $E$.

**Example 16:** *Consider the following boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + u^5(t) + \frac{1}{3} = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} \left(u^5(s) + \frac{1}{2\sqrt[3]{x}}\right) ds. \end{cases} \qquad (P_2)$$

*Let,*

$$\alpha = \frac{7}{2}, \ \beta = \frac{3}{2},$$

*And,*

$$f(t,u(t)) = u^5(t) + \frac{1}{3}, \quad g(t,u(t)) = u^5(s) + \frac{1}{2\sqrt[3]{x}}.$$

*Obviously,* $f,g\in C((0,1)\times(0,+\infty),[0,+\infty))$, $\Gamma(\alpha)=\Gamma(\frac{7}{2})\approx 3,32$, $\Gamma(\alpha-\beta)=\Gamma(2)=1$. *For any positive numbers* $r_1<r_2$, *it is easy to see that* $(Q_1)-(Q_3)$ *hold for* $p_{r_1,r_2}(t)=r_2^2+\frac{1}{2}t^{-\frac{5}{6}}r_1^{-\frac{1}{3}}$. *The height functions* $\varphi(t,r)$ *and* $\psi(t,r)$ *satisfy the following inequality:*

$$\varphi(t,r)=\max\left\{u^5(t)+\frac{1}{3}\ :\ t^{\frac{7}{2}}r\leq u\leq r\right\}\leq r^5+\frac{1}{2}t^{-\frac{5}{6}}r^{-\frac{1}{3}},$$

$$\psi(t,r)=\min\left\{u^5(t)+\frac{1}{3}\ :\ t^{\frac{7}{2}}r\leq u\leq r\right\}\geq t^{\frac{25}{2}}r^5+\frac{1}{2}r^{-\frac{1}{3}},$$

*and* $g(t,u)$ *is nondecreasing on* $u$ *for any* $t\in[0,1]$.

*It follows that*

$$\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\varphi(s,1)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds<1.$$

*And*

$$\frac{1}{\Gamma(\alpha)}\int_0^1 G_1(s)\psi\left(s,\frac{1}{125}\right)ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 G_1(s)g(s,u(s))ds>\frac{1}{100}.$$

*By Theorem* 14, *we get that* $(P_2)$ *has at least one strictly increasing positive solution* $u^*\in K$ *such that* $\frac{1}{125}\leq\|u^*\|\leq 1$.

## Conclusion

In this paper, we considered a fractional differential equation involving Riemann-Liouville fractional derivative of order $\alpha$, $2<\alpha\leq 3$. We studied the uniqueness of solution by using the Banach contraction principle and, we established the existence of positive solution by Guo-Krasnosel'skii fixed point theorem by introducing height functions of the nonlinear term on some bounded sets and considering integrations of these height functions. As application, examples are presented to illustrate the main results.

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